9/5 Recitation Solutions

In this recitation, we will use the commands found in the **Julia4Systems.ipynb** notebook.

1. Use Gaussian elimination to put each system in upper triangular form and determine how many solutions each systems has. If a system has a unique solution, use back substitution to find it and verify your calculation with Julia by solving the relevant matrix equation $A\mathbf{x} = \mathbf{b}$ (beware of rounding errors). What is happening geometrically in each system?

a) $\begin{aligned} x + 2y - 2z &= -15 \\ 2x + y - 5z &= -21 \\ x - 4y + z &= 18 \end{aligned}$ b) $\begin{aligned} x + 2y - 2z &= -15 \\ 2x + y - 5z &= -21 \\ -x - 2y + 2z &= 14 \end{aligned}$ c) $\begin{aligned} x + 2y - 2z &= -15 \\ 2x + y - 5z &= -21 \\ -x - 2y + 2z &= 15 \end{aligned}$

Part a)

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 2 & -2 & -15 \\ 2 & 1 & -5 & -21 \\ 1 & -4 & 1 & 18 \end{array}\right] \xrightarrow[\text{Replace R3 with R3-R1}]{\text{Replace R2 with R2-2R1}} \left[\begin{array}{ccc|c} 1 & 2 & -2 & -15 \\ 0 & -3 & -1 & 9 \\ 0 & -6 & 3 & 33 \end{array}\right]$$

$$\xrightarrow{\text{Replace R3 with R3-2R2}} \left[\begin{array}{ccc|c} 1 & 2 & -2 & -15 \\ 0 & -3 & -1 & 9 \\ 0 & 0 & 5 & 15 \end{array}\right]$$

This gives the system

$$\begin{aligned} x + 2y - 2z &= -15 \\ -3y - z &= 9 \\ 5z &= 15 \end{aligned}$$

Backward substitution gives the solution $x = -1, y = -4, z = 3$. Now to compute this in Julia, we use the following code:

Code:

```
A = [1 2 -2 ; 2 1 -5; 1 -4 1];
b = [-15 ; -21; 18];
x = A\ b
```

Geometrically, each equation describes a plane in $\mathbb{R}^3$. Since there is a unique solution, these three planes are intersecting simultaneously in a single point.

Part b)

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 2 & -2 & -15 \\ 2 & 1 & -5 & -21 \\ -1 & -2 & 2 & 14 \end{array}\right] \xrightarrow[\text{Replace R3 with R3+R1}]{\text{Replace R2 with R2-2R1}} \left[\begin{array}{ccc|c} 1 & 2 & -2 & -15 \\ 0 & -3 & -1 & 9 \\ 0 & 0 & 0 & -1 \end{array}\right]$$

Since there is a pivot in the final column, there is no solution (the final row corresponds to the equation $0 = -1$).

Geometrically, the planes corresponding to the first and last equations are parallel; hence there is no solution.

Part c)

$$[A|\mathbf{b}] = \left[\begin{array}{ccc|c} 1 & 2 & -2 & -15 \\ 2 & 1 & -5 & -21 \\ -1 & -2 & 2 & 14 \end{array}\right] \xrightarrow[\text{Replace R3 with R3+R1}]{\text{Replace R2 with R2-2R1}} \left[\begin{array}{ccc|c} 1 & 2 & -2 & -15 \\ 0 & -3 & -1 & 9 \\ 0 & 0 & 0 & 0 \end{array}\right]$$

Since there is no pivot in the last column and there is a row with no pivot, this system has infinitely many solutions. Geometrically, the planes given by the first and last equations are identical (since the equations are multiples of each other). Now the intersection between this plane and the plane given by the second equation is a line, which provides infinitely many solutions.

2. For what values of k does the following matrix have three pivots?

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 2 & 2k \end{bmatrix}$$

$$\begin{bmatrix} 2 & 1 & 0 \\ 3 & 2 & 1 \\ 2 & 2 & 2k \end{bmatrix} \xrightarrow[\text{Replace R3 with R3-R1}]{\text{Replace R2 with R2-}\frac{3}{2}\text{R1}} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 1 & 2k \end{bmatrix} \xrightarrow{\text{Replace R3 with R3-2R2}} \begin{bmatrix} 2 & 1 & 0 \\ 0 & \frac{1}{2} & 1 \\ 0 & 0 & 2k-2 \end{bmatrix}$$

There are three pivot positions exactly when $2k - 2 \neq 0$, or $k \neq 1$.

3. Which of the following matrices are in RREF? For the matrices not in RREF, describe the row operations needed to put them in RREF.

$$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 3 & 0 \\ 0 & 1 & 4 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 & 3 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 2 \end{bmatrix} \quad \begin{bmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 1 \end{bmatrix}$$

The second and third matrices are in RREF. The first is not since the the (1,2)-entry is nonzero, although it is above a pivot, and the third row contains all zeros and is not at the bottom; to put it into RREF, we need to replace R1 with R1-R2 and then swap R3 and R4. The fourth matrix is also not in RREF since the second pivot entry is -1; to put it into RREF, we need to mutiply R2 by -1. The fifth is not in RREF since the pivot in the second row is to the right of the pivot in the third row; to put it in RREF, we can first swap R2 and R3, then replace R2 with R2+R3, and finally multiply R3 by -1.

Looking Ahead:
Let $A = \begin{bmatrix} 1 & 2 \\ 1 & 1 \end{bmatrix}$. Solve the matrix equations $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ and $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$, where $\mathbf{x} = \begin{bmatrix} x \\ y \end{bmatrix}$. Let $\mathbf{a}$ be the solution to the first equation and let $\mathbf{b}$ be the solution to the second equation. Let $B = \begin{bmatrix} \mathbf{a} & \mathbf{b} \end{bmatrix}$ be the 2×2 matrix whose columns are the solutions to the above matrix equations. Show that $AB = I$ and $BA = I$.[1]

The matrix equation $A\mathbf{x} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}$ is equivalent to solving the system $x + 2y = 1$, $x + y = 0$. Elimination and backward substitution gives the solution $x = -1, y = 1$; hence $\mathbf{a} = \begin{bmatrix} -1 \\ 1 \end{bmatrix}$. Similarly, $A\mathbf{x} = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$ is equivalent to solving the system $x + 2y = 0$, $x + y = 1$. Elimination and backward substitution gives the solution $x = 2, y = -1$; hence $\mathbf{b} = \begin{bmatrix} 2 \\ -1 \end{bmatrix}$.

Now $B = \begin{bmatrix} -1 & 2 \\ 1 & -1 \end{bmatrix}$. A quick calculation shows that $AB = BA = I$.

4. **Just For Fun**: Geometry of Higher Dimensions. An equation in two variables can be represented by a line in $\mathbb{R}^2$; two such lines can intersect in a point. An equation in three variables can be represented by a plane in $\mathbb{R}^3$; two such planes can intersect in a line. How can one represent an equation in four variables? What does the intersection of two such objects look like?

   An equation in four variables can be represented by a 3-dimensional slice of $\mathbb{R}^4$; which looks like $\mathbb{R}^3$. Two such objects can intersect in a plane, which looks like $\mathbb{R}^2$.

[1] A and B are called *inverses* of each other. We will discuss inverses tomorrow.